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Excitations and the S-matrix for an $su(3)$ spin chain combining $\{3\}$ and $\{3^*\}$ representations

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Abstract. The associated Hamiltonian for an $su(3)$ spin chain combining $\{3\}$ and $\{3^*\}$ representations is calculated. The ansatz equations for this chain are obtained and solved in the thermodynamic limit, and the ground state and excitations are described. Thus, relations between the number of roots and the number of holes in each level have been found. The excited states are characterized by means of these quantum numbers. Finally, the exact S-matrix for a state with two holes is found.

1. Introduction

The Yang–Baxter equation (YBE) [1, 2] and the quantum inverse scattering method (QISM) [3] have contributed to the discovery and solution of numerous many-body quantum systems. The best known system is the Heisenberg model, which was solved by Bethe [4]. This model can be derived from the YBE using the $su(2)$ Lie algebra. Generalizations of this model have been obtained using other Lie algebras [5–7].

An interesting problem is the derivation of integrable models where the chain is formed by two kinds of state. The original work, an alternating chain with $s = 1/2$ and $s = 1$, was presented in [8]. Later, several works, using several Lie algebras, were studied [9]. In these systems it is possible to solve the ansatz equations in the thermodynamic limit [10]. This allows one to describe the system by means of quantum numbers and therefore one is able to find the ground state and the excited states [11–13]. Moreover, the S-matrix for the scattering of excitations can be determined [14–21].

In this paper we use the $su(3)$ rational solutions of the YBE and we form a chain combining $\{3\}$ and $\{3^*\}$ representations. For the alternating chain we find the Hamiltonian, which contains a coupling of three neighbouring site terms. We solve the ansatz equations and we deduce the root and hole densities. The relations between these densities allow us to describe the ground and excited states. In the last section we calculate the exact S-matrix for two-hole scattering.

2. The model and the Hamiltonian

In this section, we construct an alternating chain that mixes the $\{3\}$ and $\{3^*\}$ representations of $su(3)$. We use the rational solutions of the YBE. If we take the $\{3\}$ representation as auxiliary space and {3} as site space, we have the operator

$$L^{(\{3\},\{3\})}(u) = (1 - \mathrm{i}u) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - \mathrm{i}u \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,j} \otimes e_{k,k} + \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,k} \otimes e_{k,j}. \tag{2.1}$$

For the {3} representation as auxiliary and {3*} as site, the operator is

$$L^{(\{3\},\{3^*\})}(u) = \left(\frac{1}{2} - \mathrm{i}u\right) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - \left(\frac{3}{2} - \mathrm{i}u\right) \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,j} \otimes e_{k,k} - \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,k} \otimes e_{j,k} \tag{2.2}$$

with $(e_{l,m})_{i,j} = \delta_{l,i}\delta_{m,j}$.

We consider a chain with N sites (N even) in which the site spaces are alternating in the representations {3} and {3*}. The monodromy matrix, which describes the transportation along the chain, is defined by

$$T_{a,b}(u,\alpha) = L^{(\{3\},\{3\})}_{a,a_1}(u) L^{(\{3\},\{3^*\})}_{a_1,a_2}(u+\alpha) \ldots L^{(\{3\},\{3\})}_{a_{N-2},a_{N-1}}(u) L^{(\{3\},\{3^*\})}_{a_{N-1},b}(u+\alpha) \tag{2.3}$$

where the indices are in the auxiliary space and α is an arbitrary parameter.

Since $L^{(\{3\},\{3\})}(u)$ and $L^{(\{3\},\{3^*\})}(u)$ operators verify the YBE, then $T(u,\alpha)$ also verifies it,

$$R(u-v)\cdot(T(u,\alpha)\otimes T(v,\alpha)) = (T(v,\alpha)\otimes T(u,\alpha))\cdot R(u-v) \tag{2.4}$$

with

$$R(u) = (1 - \mathrm{i}u) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - \mathrm{i}u \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,k} \otimes e_{k,j} + \sum_{\substack{j,k=1 \\ j\neq k}}^{3} e_{j,j} \otimes e_{k,k}. \tag{2.5}$$

Following standard procedure, we take the transfer matrix as the trace, on the auxiliary space, of the monodromy matrix

$$F(u,\alpha) = \mathrm{trace}[T(u,\alpha)]. \tag{2.6}$$

Due to the YBE, the transfer matrices commute for different values of the argument

$$[F(u,\alpha), F(v,\alpha)] = 0. \tag{2.7}$$

The Hamiltonian of this system is defined by the first derivative of the transfer matrix,

$$H(\alpha) = \frac{\mathrm{d}}{\mathrm{d}u} \ln F(u,\alpha)|_{u=0}. \tag{2.8}$$

Collecting the diverse terms, the Hamiltonian becomes

$$H(\alpha) = \frac{\mathrm{i}}{\bar{\rho}(\alpha)} \sum_{\substack{j=1 \\ j\ \mathrm{odd}}}^{N-1} h^{[1]}_{j,j+1} + \frac{\mathrm{i}}{c_1\bar{\rho}(\alpha)} \sum_{\substack{j=1 \\ j\ \mathrm{odd}}}^{N-1} h^{[2]}_{j,j+1,j+2} \tag{2.9}$$

with

$$(h^{[1]}_{j,j+1})_{a,b;\beta,\gamma} = [\dot{L}^{(\{3\},\{3^*\})}_{a,c}(\alpha)]_{\beta,\delta}[L^{(\{3^*\},\{3\})}_{\delta,\gamma}(-\alpha)]_{c,b} \tag{2.10a}$$

$$(h^{[2]}_{j,j+1,j+2})_{a,b;\beta,\gamma;c,d} = [L^{(\{3\},\{3^*\})}_{a,e}(\alpha)]_{\beta,\delta}[\dot{L}^{(\{3\},\{3\})}_{e,d}(0)]_{c,f}[L^{(\{3^*\},\{3\})}_{\delta,\gamma}(-\alpha)]_{f,b} \tag{2.10b}$$

and

$$R(0) = c_1 I \tag{2.11a}$$

$$[L^{(\{3\},\{3^*\})}_{a,b}(u)]_{\alpha,\beta}[L^{(\{3^*\},\{3\})}_{\beta,\gamma}(-u)]_{b,c} = \bar{\rho}(u)\delta_{a,c}\delta_{\alpha,\gamma}. \tag{2.11b}$$

Thus, we find

$$H(\alpha) = \frac{2}{9+4\alpha^2}\left\{ \sum_{\substack{i=1 \\ i\ \mathrm{odd}}}^{N-1} \sum_{a=1}^{8} \lambda_i^a \otimes \bar{\lambda}_{i+1}^a + \sum_{\substack{i=2 \\ i\ \mathrm{even}}}^{N} \sum_{a=1}^{8} \bar{\lambda}_i^a \otimes \lambda_{i+1}^a \right.$$
$$+ \sum_{\substack{i=1 \\ i\ \mathrm{odd}}}^{N-1} \sum_{a,b,c=1}^{8} \left(\frac{3}{2} d_{a,b,c} - \alpha f_{a,b,c} \right) \lambda_i^a \otimes \bar{\lambda}_{i+1}^b \otimes \lambda_{i+2}^c$$
$$\left. + \frac{5+4\alpha^2}{4} \sum_{\substack{i=1 \\ i\ \mathrm{odd}}}^{N-1} \sum_{a=1}^{8} \lambda_i^a \otimes I \otimes \lambda_{i+2}^a \right\} + \frac{41+12\alpha(\alpha+\mathrm{i})}{9(9+4\alpha^2)} I \qquad (2.12)$$

where we have used the Gell–Mann matrices λ and $\bar{\lambda}$ for the $\{3\}$ and $\{3^*\}$ representations, respectively, $d_{a,b,c}$ and $f_{a,b,c}$ being the structure constants of $SU(3)$. When $\alpha = 0$, we obtain the simplest case.

3. Diagonalization and ansatz equations

We have solved the chain that mixes the $\{3\}$ and $\{3^*\}$ representations of $su(3)$, using the method given in [9]. The eigenvalue of the transfer matrix is

$$\Lambda(u) = [a(u)]^{N_3} [\bar{b}(u)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - u) + [b(u)]^{N_3} \prod_{i=1}^{r} g(u - \mu_i)$$
$$\times \left\{ [\bar{b}(u)]^{N_3^*} \prod_{l=1}^{s} g(\lambda_l - u) + [\bar{a}(u)]^{N_3^*} \prod_{k=1}^{s} g(u - \lambda_k) \prod_{n=1}^{r} \frac{1}{g(u-\mu_n)} \right\} \qquad (3.1)$$

and the coupled Bethe equations are

$$[g(\mu_k)]^{N_3} = \prod_{\substack{j=1 \\ j\neq k}}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) \qquad (3.2a)$$

$$[\bar{g}(\lambda_l)]^{N_3^*} = \prod_{j=1}^{r} g(\lambda_l - \mu_j) \prod_{\substack{i=1 \\ i\neq l}}^{s} \frac{g(\lambda_i - \lambda_l)}{g(\lambda_l - \lambda_i)}$$
$$k = 1, \ldots, r \qquad l = 1, \ldots, s \qquad (3.2b)$$

with

$$a(u) = 1 - \mathrm{i}u \qquad (3.3a)$$
$$b(u) = -\mathrm{i}u \qquad (3.3b)$$
$$\bar{a}(u) = \frac{1}{2} - \mathrm{i}u \qquad (3.3c)$$
$$\bar{b}(u) = \frac{3}{2} - \mathrm{i}u \qquad (3.3d)$$
$$g(u) = \frac{a(u)}{b(u)} \qquad (3.3e)$$
$$\bar{g}(u) = \frac{\bar{a}(u)}{\bar{b}(u)}. \qquad (3.3f)$$

It is convenient to set the parametrization

$$\mu_j = v_j^{(1)} - \frac{\mathrm{i}}{2} \tag{3.4a}$$

$$\lambda_j = v_j^{(2)} - \mathrm{i}. \tag{3.4b}$$

Using such a parametrization, the Bethe equations can be written as

$$\left[\frac{v_k^{(1)} - \mathrm{i}/2}{v_k^{(1)} + \mathrm{i}/2}\right]^{N_3} = -\prod_{j=1}^{r} \frac{v_k^{(1)} - v_j^{(1)} - \mathrm{i}}{v_k^{(1)} - v_j^{(1)} + \mathrm{i}} \prod_{l=1}^{s} \frac{v_l^{(2)} - v_k^{(1)} - \mathrm{i}/2}{v_l^{(2)} - v_k^{(1)} + \mathrm{i}/2} \tag{3.5a}$$

$$\left[\frac{v_k^{(2)} + \mathrm{i}/2}{v_k^{(2)} - \mathrm{i}/2}\right]^{N_3^*} = -\prod_{j=1}^{r} \frac{v_k^{(2)} - v_j^{(1)} - \mathrm{i}/2}{v_k^{(2)} - v_j^{(1)} + \mathrm{i}/2} \prod_{l=1}^{s} \frac{v_l^{(2)} - v_k^{(2)} - \mathrm{i}}{v_l^{(2)} - v_k^{(2)} + \mathrm{i}}. \tag{3.5b}$$

We define the function

$$\phi(x) = \ln \frac{1 + \mathrm{i}x}{1 - \mathrm{i}x} \equiv 2\mathrm{i} \arctan x \tag{3.6}$$

and taking logarithms in (3.5a) and (3.5b) we obtain

$$N_3\phi(2v_k^{(1)}) - \sum_{j=1}^{r} \phi(v_k^{(1)} - v_j^{(1)}) + \sum_{l=1}^{s} \phi(2v_k^{(1)} - 2v_l^{(2)}) = 2\pi I_k^{(1)} \qquad 1 \leqslant k \leqslant r \tag{3.7a}$$

$$N_3^*\phi(2v_k^{(2)}) + \sum_{j=1}^{r} \phi(2v_k^{(2)} - 2v_j^{(1)}) - \sum_{l=1}^{s} \phi(v_k^{(2)} - v_l^{(2)}, \gamma) = 2\pi I_k^{(2)} \qquad 1 \leqslant k \leqslant s \tag{3.7b}$$

where $I_k^{(1)}$ and $I_k^{(2)}$ are half-integers.

In the thermodynamic limit $N \to \infty$, the roots tend to have continuous distributions. Unlike what happens in other cases, we cannot distinguish between the roots coming from the different types of representations; this is noted by simple inspection of the equations of the ansatz. Therefore, we define two root densities, one for each level,

$$\rho_l(v_j^{(l)}) = \lim_{N_3 \to \infty} \frac{1}{N_3(v_{j+1}^{(l)} - v_j^{(l)})} \qquad l = 1, 2. \tag{3.8}$$

Let

$$Z_{N_3}(v) = \frac{1}{2\pi}\left[\phi(2v) - \frac{1}{N_3}\sum_{j=1}^{r} \phi(v - v_j^{(1)}) + \frac{1}{N_3}\sum_{j=1}^{s} \phi(2v - 2v_j^{(2)})\right] \tag{3.9a}$$

$$Z_{N_3^*}(v) = \frac{1}{2\pi}\left[\phi(2v) - \frac{1}{N_3^*}\sum_{j=1}^{s} \phi(v - v_j^{(2)}) + \frac{1}{N_3^*}\sum_{j=1}^{r} \phi(2v - 2v_j^{(1)})\right]. \tag{3.9b}$$

In the thermodynamic limit, the derivative of these functions are

$$\sigma_1(v) \equiv \frac{\mathrm{d}}{\mathrm{d}v} Z_{N_3}(v) \approx \frac{N}{N_3}\rho_1(v) + \frac{1}{N_3}\sum_{h=1}^{N_h^{(1)}} \delta(v - \theta_h^1) \tag{3.10a}$$

$$\sigma_2(v) \equiv \frac{\mathrm{d}}{\mathrm{d}v} Z_{N_3^*}(v) = \frac{N}{N_3^*}\rho_2(v) + \frac{1}{N_3^*}\sum_{h=1}^{N_h^{(2)}} \delta(v - \theta_h^2). \tag{3.10b}$$

Using the approximation

$$\lim_{N \to \infty} \frac{1}{N} \sum_j f(v_j^{(k)}) \simeq \int \mathrm{d}\lambda\, f(\lambda)\rho_k(\lambda) \tag{3.11}$$

and performing the Fourier transform, we can solve the system of equations. Thus, we write

$$\sigma_1(v) = \sigma_1^{(o)}(v) + \frac{1}{N_3}\sigma_1^{(h)}(v) \tag{3.12a}$$

$$\sigma_2(v) = \sigma_2^{(o)}(v) + \frac{1}{N_3^*}\sigma_2^{(h)}(v) \tag{3.12b}$$

where $\sigma_k^{(o)}(v)$ and $\sigma_k^{(h)}(v)$ show the root contribution and hole contribution, respectively, for the k-level. One finds

$$\sigma_1^{(o)}(v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\sinh\alpha}{\sinh(3\alpha/2)} + \frac{N_3^*}{N_3}\frac{\sinh(\alpha/2)}{\sinh(3\alpha/2)}\right)\mathrm{e}^{\mathrm{i}\alpha v}\,\mathrm{d}\alpha \tag{3.13a}$$

$$\sigma_2^{(o)}(v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\sinh\alpha}{\sinh(3\alpha/2)} + \frac{N_3}{N_3^*}\frac{\sinh(\alpha/2)}{\sinh(3\alpha/2)}\right)\mathrm{e}^{\mathrm{i}\alpha v}\,\mathrm{d}\alpha \tag{3.13b}$$

$$\sigma_1^{(h)}(v) = \frac{1}{2\pi}\left\{\sum_{h=1}^{N_h^{(1)}} r_a(v-\theta_h^{(1)}) - \sum_{h=1}^{N_h^{(2)}} r_b(v-\theta_h^{(2)})\right\} \tag{3.13c}$$

$$\sigma_2^{(h)}(v) = \frac{1}{2\pi}\left\{\sum_{h=1}^{N_h^{(2)}} r_a(v-\theta_h^{(2)}) - \sum_{h=1}^{N_h^{(1)}} r_b(v-\theta_h^{(1)})\right\} \tag{3.13d}$$

with

$$r_a(x) = \int_{-\infty}^{+\infty}\frac{\sinh(\alpha/2)}{\sinh(3\alpha/2)}\,\mathrm{e}^{\mathrm{i}\alpha x-|\alpha|}\,\mathrm{d}\alpha \tag{3.14a}$$

$$r_b(x) = \int_{-\infty}^{+\infty}\frac{\sinh(\alpha/2)}{\sinh(3\alpha/2)}\,\mathrm{e}^{\mathrm{i}\alpha x+|\alpha|/2}\,\mathrm{d}\alpha. \tag{3.14b}$$

4. Ground state and excitations

Any physical state is characterized by two sets of roots satisfying the Bethe equations (3.5a) and (3.5b). These roots can be complex and, moreover, we can find some modifications of the distribution of roots (holes). So, we write the solutions of the Bethe equations as strings

$$v_{k,(m)}^{(1)} = v_{k,M}^{(1)} + \mathrm{i}m \qquad m = -M,\ldots,+M \tag{4.1a}$$

$$v_{k,(m)}^{(2)} = v_{k,M'}^{(2)} + \mathrm{i}m \qquad m = -M',\ldots,+M'. \tag{4.1b}$$

An M-string has length M and contains M roots, which share the same real part. The 0-strings are real numbers. In order to find the equations for the centre of strings we introduce (4.1a) and (4.1b) into (3.5a) and (3.5b) and multiply the Bethe equations for all the roots of the same string. Using the appendix, we obtain

$$2N_3\arctan\frac{v_{k,M}^{(1)}}{M+\frac{1}{2}} = 2\pi Q_{k,M}^{(1)} + \sum_{M'}\sum_{j=1}^{\nu_{M'}^{(1)}}\psi_{M,M'}(v_{k,M}^{(1)} - v_{j,M'}^{(1)}) - \sum_{M''}\sum_{l=1}^{\nu_{M''}^{(2)}}\phi_{M,M''}(v_{k,M}^{(1)} - v_{l,M''}^{(2)}) \tag{4.2a}$$

$$-2N_3^*\arctan\frac{v_{k,M}^{(2)}}{M+\frac{1}{2}} = -2\pi Q_{k,M}^{(2)} - \sum_{M'}\sum_{j=1}^{\nu_{M'}^{(2)}}\psi_{M,M'}(v_{k,M}^{(2)} - v_{j,M'}^{(2)})$$

$$+\sum_{M''}\sum_{l=1}^{\nu_{M''}^{(1)}}\phi_{M,M''}(v_{k,M}^{(2)}-v_{l,M''}^{(1)}) \qquad (4.2b)$$

where $\nu_M^{(i)}$ is the number of M-strings at level i, and the numbers $Q_{k,M}^{(1)}$ and $Q_{k,M}^{(2)}$ are integers or half-odd. They vary in the intervals $|Q_{k,M}^{(1)}| \leqslant Q_{\max,M}^{(1)}$ and $|Q_{k,M}^{(2)}| \leqslant Q_{\max,M}^{(2)}$. In order to obtain $Q_{\max,M}^{(1)}$ and $Q_{\max,M}^{(2)}$, we define the functions

$$F_M^{(1)}(\lambda) = \frac{N_3}{\pi}\arctan\frac{\lambda}{M+\frac{1}{2}} - \frac{1}{2\pi}\sum_{M'}\sum_{j=1}^{\nu_{M'}^{(1)}}\psi_{M,M'}(\lambda - v_{j,M'}^{(1)})$$
$$+\frac{1}{2\pi}\sum_{M''}\sum_{l=1}^{\nu_{M''}^{(2)}}\phi_{M,M''}(\lambda - v_{l,M''}^{(2)}) \qquad (4.3a)$$

$$F_M^{(2)}(\lambda) = \frac{N_3^*}{\pi}\arctan\frac{\lambda}{M+\frac{1}{2}} - \frac{1}{2\pi}\sum_{M'}\sum_{j=1}^{\nu_{M'}^{(2)}}\psi_{M,M'}(\lambda - v_{j,M'}^{(2)})$$
$$+\frac{1}{2\pi}\sum_{M''}\sum_{l=1}^{\nu_{M''}^{(1)}}\phi_{M,M''}(\lambda - v_{l,M''}^{(1)}). \qquad (4.3b)$$

Equations (4.2a) and (4.2b) can be written as follows:

$$F_M^{(1)}(v_{j,M}^{(1)}) = Q_{k,M}^{(1)} \qquad (4.4a)$$
$$F_M^{(2)}(v_{j,M}^{(2)}) = Q_{k,M}^{(2)}. \qquad (4.4b)$$

Note that $F_M^{(1)}(\lambda)$ and $F_M^{(2)}(\lambda)$ are increasing functions of λ, so we deduce that

$$F_M^{(1)}(-\infty) \leqslant Q_{k,M}^{(1)} \leqslant F_M^{(1)}(+\infty) \qquad (4.5a)$$
$$F_M^{(2)}(-\infty) \leqslant Q_{k,M}^{(2)} \leqslant F_M^{(2)}(+\infty). \qquad (4.5b)$$

The total number of allowed $Q_{k,M}^{(i)}$ will be

$$2Q_{\max,M}^{(i)} + 1 = 2F_M^{(i)}(+\infty). \qquad (4.6)$$

If we denote by $H_M^{(i)}$ the number of holes in the sea of M-strings at level i, then we have

$$2Q_{\max,M}^{(i)} + 1 = \nu_M^{(i)} + H_M^{(i)} \qquad (4.7)$$

because the total number of allowed $Q_{k,M}^{(i)}$ corresponds to the sum of roots and holes.

Taking the limit when λ tends to infinity in (4.3a) and (4.3b) and using (4.6) and (4.7) we get

$$\nu_M^{(1)} + H_M^{(1)} = N_3 - 2\sum_{M'\geqslant 0} J(M,M')\nu_{M'}^{(1)} + 2\sum_{M''\geqslant 0} K(M,M'')\nu_{M''}^{(2)} \qquad (4.8a)$$
$$\nu_M^{(2)} + H_M^{(2)} = N_3^* - 2\sum_{M'\geqslant 0} J(M,M')\nu_{M'}^{(2)} + 2\sum_{M''\geqslant 0} K(M,M'')\nu_{M''}^{(1)} \qquad (4.8b)$$

with

$$J(M_1,M_2) = \begin{cases} 2M_1 + \frac{1}{2} & \text{if } M_1 = M_2 \\ 2\min(M_1,M_2)+1 & \text{if } M_1 \neq M_2 \end{cases} \qquad (4.9a)$$

$$K(M_1,M_2) = \begin{cases} M_2 + \frac{1}{2} & \text{if } M_2 + \frac{1}{2} \leqslant M_1 \\ M_1 + \frac{1}{2} & \text{if } M_2 + \frac{1}{2} > M_1. \end{cases} \qquad (4.9b)$$

If N_ρ is the number of states ρ in the chain, then, as we have shown in a recent paper [22],

$$N_{\rm u} - N_{\bar{\rm u}} = N_3 - r \qquad (4.10a)$$

$$N_{\rm d} - N_{\bar{\rm d}} = r - s \qquad (4.10b)$$

$$N_{\rm s} - N_{\bar{\rm s}} = s - N_3^*. \qquad (4.10c)$$

On the other hand, the total number of strings is r and s at the first and second levels, respectively, that is

$$r = \sum_{M\geqslant 0}(2M+1)\nu_M^{(1)} \qquad (4.11a)$$

$$s = \sum_{M\geqslant 0}(2M+1)\nu_M^{(2)} \qquad (4.11b)$$

where M is integer or half-odd. Applying (4.8a) and (4.8b) for the real roots, and using (4.9a) and (4.9b), we find

$$H_0^{(1)} = N_3 - 2\sum_{M'\geqslant 0}\nu_{M'}^{(1)} + \sum_{M''\geqslant 0}\nu_{M''}^{(2)} \qquad (4.12a)$$

$$H_0^{(2)} = N_3^* - 2\sum_{M'\geqslant 0}\nu_{M'}^{(2)} + \sum_{M''\geqslant 0}\nu_{M''}^{(1)}. \qquad (4.12b)$$

For the general case, this relation can be written as

$$\nu_n^{(1)} - \frac{\nu_n^{(2)}}{2} + H_n^{(1)} = \frac{H_{n-1/2}^{(1)} + H_{n+1/2}^{(1)}}{2} \qquad (4.13a)$$

$$\nu_n^{(2)} - \frac{\nu_n^{(1)}}{2} + H_n^{(2)} = \frac{H_{n-1/2}^{(2)} + H_{n+1/2}^{(2)}}{2}$$

$$n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \qquad (4.13b)$$

where we have used $H_{-1/2}^{(1)} \equiv N_3$ and $H_{-1/2}^{(2)} \equiv N_3^*$.

Now, we can characterize the ground state and the excited states.

(i) *Ground state*. In the ground state we have no holes and only real roots. That is

$$\nu_0^{(1)} = \frac{2N_3 + N_3^*}{3} \qquad (4.14a)$$

$$\nu_0^{(2)} = \frac{N_3 + 2N_3^*}{3} \qquad (4.14b)$$

$$\nu_{M>0}^{(1)} = \nu_{M>0}^{(2)} = H_{M\geqslant 0}^{(1)} = H_{M\geqslant 0}^{(2)} = 0 \qquad (4.14c)$$

and the quantum numbers are

$$N_{\rm u} - N_{\bar{\rm u}} = N_{\rm d} - N_{\bar{\rm d}} = N_{\rm s} - N_{\bar{\rm s}} = \frac{N_3 - N_3^*}{3}. \qquad (4.15)$$

Then, the ground state is formed by pairs u$\bar{\rm u}$, d$\bar{\rm d}$ and s$\bar{\rm s}$.

(ii) *Excited state with the same quantum numbers*. This is characterized by one hole and one two-string in each level,

$$\nu_0^{(1)} = \frac{2N_3 + N_3^*}{3} - 2 \qquad (4.16a)$$

$$\nu_0^{(2)} = \frac{N_3 + 2N_3^*}{3} - 2 \qquad (4.16b)$$

$$\nu_{1/2}^{(1)} = \nu_{1/2}^{(2)} = H_0^{(1)} = H_0^{(2)} = 1 \qquad (4.16c)$$

$$\nu_{M>1/2}^{(1)} = \nu_{M>1/2}^{(2)} = H_{M>0}^{(1)} = H_{M>0}^{(2)} = 0 \qquad (4.16d)$$

and the quantum numbers are as in (4.15).

(iii) *Excited state with other quantum numbers*. We can find an excited state characterized by two holes, one in each level, and real numbers, that is

$$\nu_0^{(1)} = \frac{2N_3 + N_3^*}{3} - 1 \qquad (4.17a)$$

$$\nu_0^{(2)} = \frac{N_3 + 2N_3^*}{3} - 1 \qquad (4.17b)$$

$$H_0^{(1)} = H_0^{(2)} = 1 \qquad (4.17c)$$

$$\nu_{M>0}^{(1)} = \nu_{M>0}^{(2)} = H_{M>0}^{(1)} = H_{M>0}^{(2)} = 0. \qquad (4.17d)$$

Here, the quantum numbers are

$$N_{\mathrm{u}} - N_{\bar{\mathrm{u}}} = \frac{N_3 - N_3^*}{3} + 1 \qquad (4.18a)$$

$$N_{\mathrm{d}} - N_{\bar{\mathrm{d}}} = \frac{N_3 - N_3^*}{3} \qquad (4.18b)$$

$$N_{\mathrm{s}} - N_{\bar{\mathrm{s}}} = \frac{N_3 - N_3^*}{3} - 1. \qquad (4.18c)$$

There are four ways to obtain this state from the ground state: the first is when a d-site changes to a u-site and a $\bar{\mathrm{d}}$-site changes to an $\bar{\mathrm{s}}$-site; the second when we have a change from a $\bar{\mathrm{u}}$ to a $\bar{\mathrm{d}}$-site and an s- to a d-site; the third is a change from an s to a u, and the fourth is a change from a $\bar{\mathrm{u}}$ to an $\bar{\mathrm{s}}$.

The next excited states have more that two holes, and they can be found by using (4.11)–(4.13).

5. *S*-matrix

In this section we are going to find the S-matrix for the excitations over the ground state. For this purpose we will use the transfer matrix $\bar{F}(u,\gamma)$, which is obtained by taking the $\{3^*\}$ representation in the auxiliary space. Thus, we define

$$\bar{T}_{\alpha,\beta}(u,\gamma) = L_{\alpha,\alpha_1}^{(\{3^*\},\{3\})}(u+\gamma) L_{\alpha_1,\alpha_2}^{(\{3^*\},\{3^*\})}(u) \ldots L_{\alpha_{N-2},\alpha_{N-1}}^{(\{3^*\},\{3\})}(u+\gamma) L_{\alpha_{N-1},\beta}^{(\{3^*\},\{3^*\})}(u) \qquad (5.1)$$

and the transfer matrix is

$$\bar{F}(u,\gamma) = \mathrm{trace}[\bar{T}(u,\gamma)]. \qquad (5.2)$$

From the YBE we have that

$$[T(u,\gamma), \bar{T}(v,-\gamma)] = 0. \qquad (5.3)$$

We take the simplest case, that is $\gamma = 0$.

The eigenvalue of $\bar{F}(u,0)$ is

$$\bar{\Lambda}(u) = [a(u)]^{N_3^*}[\bar{b}(u)]^{N_3} \prod_{j=1}^{\bar{r}} g(\bar{\mu}_j - u) + [b(u)]^{N_3^*} \prod_{i=1}^{\bar{r}} g(u - \bar{\mu}_i)$$
$$\times \left\{ [\bar{b}(u)]^{N_3} \prod_{l=1}^{\bar{s}} g(\bar{\lambda}_l - u) + [\bar{a}(u)]^{N_3} \prod_{k=1}^{\bar{s}} g(u - \bar{\lambda}_k) \prod_{n=1}^{\bar{r}} \frac{1}{g(u - \bar{\mu}_n)} \right\} \qquad (5.4)$$

and, using the parametrization (3.4), the ansatz equations can be written as

$$\left[\frac{\bar{v}_k^{(1)} - \mathrm{i}/2}{\bar{v}_k^{(1)} + \mathrm{i}/2}\right]^{N_3^*} = -\prod_{j=1}^{\bar{r}} \frac{\bar{v}_k^{(1)} - \bar{v}_j^{(1)} - \mathrm{i}}{\bar{v}_k^{(1)} - \bar{v}_j^{(1)} + \mathrm{i}} \prod_{l=1}^{\bar{s}} \frac{\bar{v}_l^{(2)} - \bar{v}_k^{(1)} - \mathrm{i}/2}{\bar{v}_l^{(2)} - \bar{v}_k^{(1)} + \mathrm{i}/2} \tag{5.5a}$$

$$\left[\frac{\bar{v}_k^{(2)} + \mathrm{i}/2}{\bar{v}_k^{(2)} - \mathrm{i}/2}\right]^{N_3} = -\prod_{j=1}^{\bar{r}} \frac{\bar{v}_k^{(2)} - \bar{v}_j^{(1)} - \mathrm{i}/2}{\bar{v}_k^{(2)} - \bar{v}_j^{(1)} + \mathrm{i}/2} \prod_{l=1}^{\bar{s}} \frac{\bar{v}_l^{(2)} - \bar{v}_k^{(2)} - \mathrm{i}}{\bar{v}_l^{(2)} - \bar{v}_k^{(2)} + \mathrm{i}}. \tag{5.5b}$$

However, the commutation of the transfer matrices in (5.3) requires that the Bethe equations (3.5) and (5.5) are the same. Thus, we have

$$\bar{v}_j^{(1)} = v_j^{(2)} \qquad j = 1, \ldots, \bar{r} = s \tag{5.6a}$$

$$\bar{v}_k^{(2)} = v_k^{(1)} \qquad k = 1, \ldots, \bar{s} = r. \tag{5.6b}$$

In order to calculate the momentum of the chain, we consider an alternating chain, that is $N_3 = N_3^* = N/2$. Then, the momentum is

$$P = \mathrm{i} \ln[\bar{\rho}(0)^{-N/2} \Lambda(0) \bar{\Lambda}(0)]. \tag{5.7}$$

With (3.1), (3.4*a*), (3.4*b*), (5.4) and (5.6*a*), (5.6*b*) we have

$$P = \mathrm{i} \sum_{j=1}^{r} \ln \frac{v_j^{(1)} - \mathrm{i}/2}{v_j^{(1)} + \mathrm{i}/2} + \mathrm{i} \sum_{k=1}^{s} \ln \frac{v_k^{(2)} - \mathrm{i}/2}{v_k^{(2)} + \mathrm{i}/2}. \tag{5.8}$$

Using the approximation (3.11) and the root densities (3.13), we get

$$P = P_0 + \sum_{h=1}^{N_h^{(1)}} p(\theta_h^{(1)}) + \sum_{h=1}^{N_h^{(2)}} p(\theta_h^{(2)}) \tag{5.9}$$

where P_0 is the momentum of the ground state and the other terms are the hole contributions, with

$$p(\theta) = \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\sinh x + \sinh(x/2)}{\mathrm{i}x \sinh(3x/2)} \, \mathrm{e}^{\mathrm{i}x\theta}. \tag{5.10}$$

We calculate the S-matrix for the scattering of two holes. Here we follow the Korepin–Andrei–Destri method [23–25]. For a state of two holes with rapidities θ_1 and θ_2, the momentum $p(\theta_1)$ verifies the quantization condition

$$\mathrm{e}^{\mathrm{i}p(\theta_1)N} S = 1. \tag{5.11}$$

The S-matrix can be written as $S = \mathrm{e}^{\mathrm{i}\Phi}$, so we have

$$p(\theta_1) + \frac{1}{N}\Phi = \frac{2\pi}{N} n \tag{5.12}$$

where n is an integer.

One can prove by direct calculation that

$$p(\theta) = \pi \int_{-\infty}^{\theta} \sigma_1^{(o)}(\lambda) \, \mathrm{d}\lambda + c_1 \tag{5.13}$$

where c_1 is a constant, which will be irrelevant for our problem. From (3.10*a*) we can write

$$Z_{N/2}(\theta) = \int_{-\infty}^{\theta} \sigma_1(\lambda) \, \mathrm{d}\lambda + c_2 \tag{5.14}$$

where c_2 is an irrelevant constant.

Evaluating for a hole in the first level θ_1 we have

$$Z_{N/2}(\theta_1) = \frac{I^{(h)}}{N/2}. \tag{5.15}$$

With the relations (5.13)–(5.15) we deduce

$$p(\theta_1) = \frac{2\pi}{N} I^{(h)} - \frac{2\pi}{N} \int_{-\infty}^{\theta_1} \sigma_1^{(h)}(\lambda)\,\mathrm{d}\lambda + \text{constant}. \tag{5.16}$$

Comparing (5.12) with (5.16), we conclude that

$$\Phi = 2\pi \int_{-\infty}^{\theta_1} \sigma_1^{(h)}(\lambda)\,\mathrm{d}\lambda + \text{constant}. \tag{5.17}$$

We remove the constants because they contribute as a rapidity-independent phase factor. Thus we have to calculate

$$\Phi(\theta) = -2 \int_0^{\infty} \frac{\mathrm{d}\alpha}{\alpha} \frac{\sinh(\alpha/2)}{\sinh(3\alpha/2)}\, \mathrm{e}^{\alpha/2} \sin(\alpha\theta) \tag{5.18}$$

where $\theta = \theta_1 - \theta_2$ is the difference of rapidities of the two holes. This integral can be solved by means of the Ψ-function

$$\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\Gamma(x) \tag{5.19}$$

and we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\theta} = \frac{1}{3}\left[-\Psi\left(\frac{1}{6} - \mathrm{i}\frac{\theta}{3}\right) + \Psi\left(\frac{1}{2} - \mathrm{i}\frac{\theta}{3}\right) - \Psi\left(\frac{1}{6} + \mathrm{i}\frac{\theta}{3}\right) + \Psi\left(\frac{1}{2} + \mathrm{i}\frac{\theta}{3}\right)\right]. \tag{5.20}$$

For the corresponding S-matrix we have

$$S_1(\theta) = \frac{\Gamma(\frac{1}{6} - \mathrm{i}\theta/3)\Gamma(\frac{1}{2} + \mathrm{i}\theta/3)}{\Gamma(\frac{1}{6} + \mathrm{i}\theta/3)\Gamma(\frac{1}{2} - \mathrm{i}\theta/3)}. \tag{5.21}$$

For the state with one hole and one two-string in each level we find the scattering matrix to be

$$S_2(\theta) = \frac{(\frac{1}{2} - \mathrm{i}\theta)}{(\frac{1}{2} + \mathrm{i}\theta)} \frac{\Gamma(\frac{1}{6} - \mathrm{i}\theta/3)\Gamma(\frac{1}{2} + \mathrm{i}\theta/3)}{\Gamma(\frac{1}{6} + \mathrm{i}\theta/3)\Gamma(\frac{1}{2} - \mathrm{i}\theta/3)}. \tag{5.22}$$

In order to calculate the S-matrix for holes in the same level we consider states with at least four holes (two in each level). Following the same procedure, we find for two holes in the first (second) level

$$S_3(\theta) = \frac{\Gamma(\frac{2}{3} - \mathrm{i}\theta/3)\Gamma(1 + \mathrm{i}\theta/3)}{\Gamma(\frac{2}{3} + \mathrm{i}\theta/3)\Gamma(1 - \mathrm{i}\theta/3)} \tag{5.23}$$

where $\theta = \theta_1^{(1)} - \theta_2^{(1)}$ ($\theta = \theta_1^{(2)} - \theta_2^{(2)}$). The S_1 and S_3 matrices coincide with those for the non-alternating chain. This shows that the scattering is the same in the alternating and the non-alternating chain [6].

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Appendix

We define the function

$$V_0(x) = \frac{x - \mathrm{i}}{x + \mathrm{i}}. \tag{A.1}$$

It is easy to prove the relations

$$\prod_{m=-M}^{M} V_0(2x + \mathrm{i}2m) = V_0\left(\frac{x}{M + \frac{1}{2}}\right) \tag{A.2a}$$

$$\prod_{m=-M}^{M} V_0(x + \mathrm{i}m) = V_0\left(\frac{x}{M}\right) V_0\left(\frac{x}{M + 1}\right) \equiv V_M(x) \tag{A.2b}$$

$$\prod_{m_1=-M_1}^{M_1} \prod_{m_2=-M_2}^{M_2} V_0(x + \mathrm{i}(m_1 + m_2)) = \prod_{L=|M_2-M_1|}^{M_1+M_2} V_L(x) \equiv V_{M_1,M_2}(x) \tag{A.2c}$$

$$\prod_{m_1=-M_1}^{M_1} \prod_{m_2=-M_2}^{M_2} V_0(2x + \mathrm{i}2(m_1 + m_2)) = \prod_{m_1=-M_1}^{M_1} V_0\left(\frac{x + \mathrm{i}m_1}{M_2 + \frac{1}{2}}\right) \equiv W_{M_1,M_2}(x). \tag{A.2d}$$

It is convenient to define the functions

$$\psi_{M_1,M_2}(x) = 2 \sum_{L=|M_2-M_1|}^{M_1+M_2} \left(\arctan\frac{x}{L} + \arctan\frac{x}{L+1}\right) \tag{A.3a}$$

$$\phi_{M_1,M_2}(x) = 2 \sum_{L=-M_1}^{M_1} \arctan\left(\frac{x + \mathrm{i}L}{M_2 + \frac{1}{2}}\right) \tag{A.3b}$$

which are connected with (A.2*c*) and (A.2*d*) by the logarithm.

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